Math 112: Introductory Real Analysis

§ Lecture on perfect sets and connected sets

Last time: compact sets

Heine-Borel theorem (characterization of compact subsets of $\mathbb{R}^k$)

Today: perfect sets and connected sets

Def A subset $E \subseteq X$ is said to be *perfect* if $E = E'$ (i.e. if every point of E is a limit point of E and if E is closed)

E.g. Every closed interval $[a,b]$ with $a<b$ is perfect.

More generally, every closed ball of positive radius in $\mathbb{R}^k$ is perfect.

~~We'll see later that there are~~

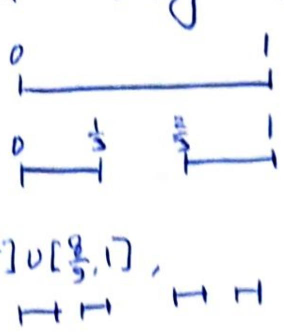
E.g. There are perfect sets in $\mathbb{R}$ which contain no segment.

For instance, let $E_0 = [0,1]$,

$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$,

$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$,

etc.



(E_{n+1} is obtained from E_n by removing the middle thirds of the intervals)

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In this way, we obtain a sequence of compact sets

$$E_0 \supset E_1 \supset E_2 \supset \cdots .$$

E_n is the union of 2^n intervals each of length $\frac{1}{3^n}$.

The set $P := \bigcap_{n=0}^{\infty} E_n$ is called the Cantor set.

Being closed and bounded, P is clearly compact.

In order to show that P is perfect, we need to show that P has no isolated points (i.e. points in P which are not limit points)

Suppose $x \in P$ were an isolated point.

Then is some open interval $B_r(x) = (x-r, x+r)$ whose intersection with P is exactly one point $\{x\}$.

Choose $n \in \mathbb{N}$ large enough so that $\frac{1}{3^n} < r$.

Let I_n be the interval of E_n containing x,

and let x_n be an end point of I_n such that $x_n \neq x$.

Then, it follows from the construction of P that $x_n \in P$.

That is, $x_n \in B_r(x) \cap P$, which contradicts our assumption that $x \in P$ is an isolated point.

Therefore, the Cantor set P is perfect.

Rmk Points in P are in 1-1 correspondence with $2^{\mathbb{N}}$, and hence P is uncountable.

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Thm Let P be a nonempty perfect set in $\mathbb{R}^k$.
Then P is uncountable.

proof) Since P has limit points, P must be infinite.
Suppose P were countable so that $P = \{x_1, x_2, x_3, \cdots\}$.
We construct a sequence $V_1 \supset V_2 \supset \cdots$ of open balls as follows.
Let $V_1 = B_R(x_1)$ be any open ball centered at x_1.
Once V_n has been constructed so that $V_n \cap P \neq \emptyset$ and

$$x_i \notin \overline{V_n} \text{ for all } 1 \leq i \leq n-1,$$

choose $x \in V_n \cap \{x_{n+1}, x_{n+2}, \cdots\}$
and $r > 0$ small enough so that $V_{n+1} := B_r(x)$ satisfies

(i) $\overline{V_{n+1}} \subset V_n$

(ii) $x_n \notin \overline{V_{n+1}}$.

Set $K_n := \overline{V_n} \cap P$. Then each K_n is non-empty, compact, and

$$K_1 \supset K_2 \supset K_3 \supset \cdots$$

Moreover, $\bigcap_{n=1}^{\infty} K_n$ must be empty since $x_n \notin K_{n+1}$ for all n.
This is a contradiction, because then $\{K_n^c\}_{n=1}^{\infty}$ would be an open cover of K_1 without any finite subcover.
Therefore, we conclude that P is uncountable $\blacksquare$

<u>Def</u> A set $E \subseteq X$ is said to be connected

if it is not the union of two disjoint non-empty subsets of E open relative to E

(i.e. if $E \not\subseteq V_1 \cup V_2$ for any disjoint open sets $V_1, V_2 \subset X$ such that $E \cap V_1 \neq \phi$ and $E \cap V_2 \neq \phi$).

E.g. $[0,1]$ is connected,

while $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ is not.

<u>Thm</u> $E \subseteq \mathbb{R}$ is connected if and only if

$[x,y] \subseteq E$ for every $x, y \in E$ with $x<y$.

proof) ($\Rightarrow$) If there exists $x, y \in E$ and $z \in (x,y)$ such that $z \notin E$,

then $E = \big(E \cap (-\infty, z)\big) \cup \big(E \cap (z, \infty)\big)$,

and hence E is not connected.

($\Leftarrow$) Suppose E is not connected. Then $E = A \cup B$ for some non-empty disjoint subsets of E that are open relative to E.

Pick $x \in A$, $y \in B$ and assume without loss of generality that $x<y$.

Set $z = \sup(A \cap [x,y])$. Then $z \notin B$, since $z \in \bar{A}$,

and
- if $z \notin A$, then $z \notin E$
- if $z \in A$, then $z \notin \bar{B}$, so there is z_1 such that $z<z_1<y$ and $z_1 \notin B$ $\Rightarrow z_1 \notin E$ ■